Mansoura University	**For all special programs**	**Final first - term exam 2015-2016**
Faculty of Engineering	**CIE, MTE, BME and BCE**	**Date: 17 - 1 - 2016 . Open Book Exam**
Physics and Eng. Math. Dept.	**Course code: Math. 106**	**Time Allowed: 2 hr. and Full Mark 50.**

BME 135
BCE 4 } 141
CIE 2

Question 1: [5 Marks] Answer by √ or ×

(1) All linear Ordinary Differential Equations are of the first degree.

(2) The two families of parabolas $y = cx^2$ and $x = cy^2$ are orthogonal.

(3) $y = e^x$ represents the solution of the BVP $y'' - y = 0,\ y(0) = 1, y(1) = 1$.

(4) If $Lf(t) = F(s)$, then $L[t\,f(t)] = sF(s) - f(0)$.

(5) The maximum rate of change of the surface $f(x, y, z) = 0$ always happens in the direction of its gradient.

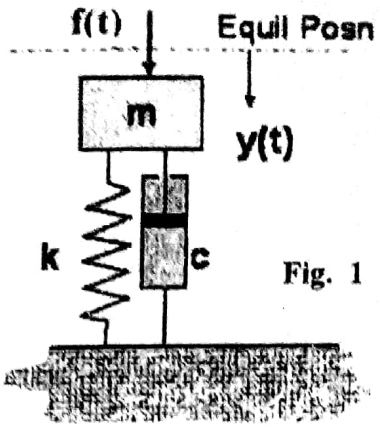

Fig. 1

Question 2 [15 Marks, each item 5 Marks]

(a) Find the orthogonal trajectory to the family of parabolas $y^2 = cx$.

(b) Find the solution to the IPV $y' = 2y + 2\sqrt{y}\,e^x,\quad y(0) = 1$.

(c) Find the general solution to the ODE $y'' - y' = e^x$.

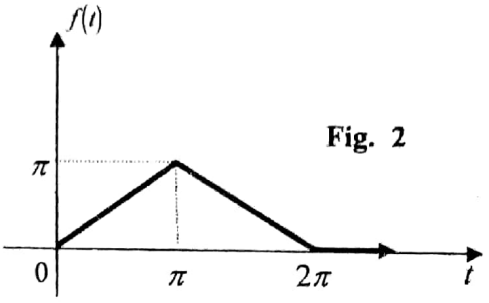

Fig. 2

Question 3 [15 Marks]

(a) Find Laplace transform to the function *f(t)* shown in **Fig. 2** and $L^{-1}\left[\dfrac{s^2+4}{s^2+9}\right]$. **[6 Marks]**

(b) Evaluate the improper integral $\int_0^\infty e^{-\sqrt{3}t}\,\dfrac{\sin t}{t}\,dt$. **[4 Marks]**

(c) In the spring mass system shown in **Fig. 1**, Find the response y(t) if m = k = 1, c = 0 and f(t) = sin(2t), assume zero initial velocity and zero initial displacement. **[5 Marks]**

Question 4 [15 Marks]

(a) Find the local extreme of the function $z = 50 + 6x + 8y - x^2 - y^2$. **[4 Marks]**

(b) Find the rate of change of the surface $x^2 + y^2 - 6x - 8y + z = 0$ at a point (*2, 2, 20*) in the direction, $i - 2j + 3k$. Find also the maximum rate of change. **[5 Marks]**

(c) Evaluate the Double integrals: (i) $\int_0^1 \int_y^1 e^{x^2}\,dx\,dy$ (ii) $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$. **[6 Marks]**

Best of luck, Prof. Ibrahim L. El-Kalla

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Model Answer

Question 1: [5 Marks] Answer by √ or ×

(1) √. **(2)** ×. **(3)** ×. **(4)** ×. **(5)** √.

Question 2 (a) [5 Marks]

<u>For the given family</u>: $2yy' = c = \frac{y^2}{x} \Rightarrow \therefore y' = \frac{y}{2x}$.

<u>For orthogonal family</u>: $y' = \frac{-2x}{y} \Rightarrow \therefore \int y\,dy = \int -2x\,dx + c$. <u>or</u>: $\frac{y^2}{2} + x^2 = c$. **Family of ellipses**

Question 2 (b) [5 Marks]

The solution of the Bernolli ODE $y' + p(x)y = q(x)y^n$ takes the form: $\mu y^{1-n} = \int (1-n)\mu q(x)dx + c$

Where the integrating factor, $\mu = Exp\left(\int (1-n)\,p(x)\,dx\right)$ **. Rewrite the ODE in the Bernolli form**

$y' - 2y = 2e^x y^{0.5}$ where, $p(x) = -2,\ q(x) = 2e^x$ and $n = 0.5$

$\Rightarrow \therefore \mu = Exp\left(\int (1-0.5)(-2)\,dx\right) = e^{-x}$ $\Rightarrow \therefore y^{0.5} = e^x\left[\int (1-0.5)2e^x e^{-x}\,dx + c\right] = e^x[x+c]$

$y(0) = 1 \Rightarrow \therefore 1 = e^0[0+c] \Rightarrow \therefore c = 1$ **Finally the solution is,** $\sqrt{y} = e^x[x+1]$

Question 2 (c) [5 Marks]

$y_G = y_c + y_p$

For $y_c \Rightarrow y'' - y' = 0 \Rightarrow \therefore c/e : m^2 - m = 0 \Rightarrow \therefore m = 0$ *or* $m = 1$

$y_c + c_1 + c_2 e^x$ **For** $y_p \Rightarrow y_p + u_1 + u_2 e^x$

Where, $u_1 = \int \frac{W_1}{W} e^x\,dx = -e^{-x}$ and $u_1 = \int \frac{W_2}{W} e^x\,dx = x$

$\therefore y_p = (x-1)e^x$ **Finally,** $y_G = c_1 + c_2 e^x + (x-1)e^x$

$$W = \begin{vmatrix} 1 & e^x \\ 0 & e^x \end{vmatrix} = e^x$$

$$W_1 = \begin{vmatrix} 0 & e^x \\ 1 & e^x \end{vmatrix} = -e^x$$

$$W_2 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

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Question 3 (a) (i)[4 Marks]

$$f(t)=\begin{cases} t & 0<t<\pi \\ 2\pi-t & \pi<t<2\pi \\ 0 & t>2\pi \end{cases}$$

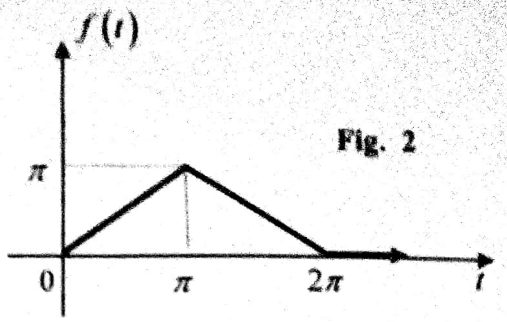


Fig. 2

$$f(t)=t[u(t)-u(t-\pi)]+(2\pi-t)[u(t-\pi)-u(t-2\pi)]+0$$

$$f(t)=t-tu(t-\pi)+(2\pi-t)u(t-\pi)-(2\pi-t)u(t-2\pi)$$

$$f(t)=t-2(t-\pi)u(t-\pi)+(t-2\pi)u(t-2\pi) \qquad \Rightarrow\therefore F(s)=\frac{1}{s^2}-2\frac{e^{-\pi s}}{s^2}+\frac{e^{-2\pi s}}{s^2}$$

Question 3 (a) (ii) [2 Marks]

$$F(s)=\frac{s^2+4}{s^2+9}=\frac{(s^2+9)-5}{s^2+9}=1-\frac{5}{s^2+9}\Rightarrow\therefore f(t)=\delta(t)-\frac{5}{3}\sin 3t$$

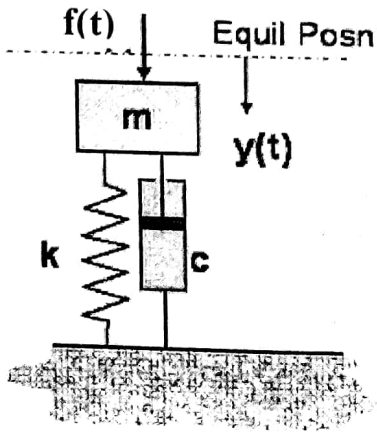
Question 3 (b) [4 Marks]

$$I=\int_0^\infty e^{-\sqrt{3}t}\left(\frac{\sin t}{t}\right)dt=L\left(\frac{\sin t}{t}\right)_{s=\sqrt{3}}$$

$$\because Lim_{t\to 0}\left(\frac{\sin t}{t}\right)=1 \text{ exist}\Rightarrow\therefore L\left(\frac{\sin t}{t}\right)=\int_s^\infty\frac{1}{x^2+1}\,dx=\left|\tan^{-1}x\right|_s^\infty=\frac{\pi}{2}-\tan^{-1}s$$

$$\therefore\ I=\left|\frac{\pi}{2}-\tan^{-1}s\right|_{s=\sqrt{3}}=\frac{\pi}{2}-\frac{\pi}{3}=\frac{\pi}{6}$$

Question 3 (c) [5 Marks]



When, m = k = 1, c = 0 and f(t) = sin(2t) then, $\ddot{y}+y=\sin 2t,\ y(0)=\dot{y}(0)=0$

Assume, *L(y)* = *Y* and the take Laplace to both sides then:

$$L\ddot{y}+Ly=L\sin 2t\Rightarrow\therefore[s^2Y-sy(0)-\dot{y}(0)]+Y=\frac{2}{s^2+4}$$

$$Y[s^2+1]=\frac{2}{s^2+4} \qquad \Rightarrow\therefore\ Y=\frac{2}{(s^2+1)(s^2+4)} \qquad \Rightarrow\therefore\ Y=\frac{2}{(s^2+1)(s^2+4)}=\frac{A}{(s^2+1)}+\frac{B}{(s^2+4)}$$

From partial fraction $A=\left|\frac{2}{(s^2+4)}\right|_{S^2=-1}=\frac{2}{3}$ and $B=\left|\frac{2}{(s^2+1)}\right|_{S^2=-4}=\frac{-2}{3}$

$$\therefore\ Y(s)=\frac{2/3}{(s^2+1)}-\frac{2/3}{(s^2+4)} \qquad \Rightarrow\therefore\ y(t)=\frac{2}{3}\sin t-\frac{1}{3}\sin 2t$$

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Question 4 (a)[4 Marks]

$z = f(x,y) = 50 + 6x + 8y - x^2 - y^2$ $\qquad f_x = 6 - 2x = 0$, $f_y = 8 - 2y = 0$ the critical point is $(3,4)$

$f_{xx} = -2$, $f_{xy} = -2$ and $f_{yy} = 0$ $\qquad \because D = f_{xx} \cdot f_{yy} - (f_{xy})^2 = 4 > 0$ and $f_{xx} < 0$

$\therefore (3,4)$ local maximum and $f_{max} = 50 + 6(3) + 8(4) - 9 - 16 = 75$

Question 4 (b)[5 Marks]

For the surface. $f(x,y,z) = x^2 + y^2 - 6x - 8y + z = 0$, $f_x = 2x - 6$, $f_y = 2y - 8$ and $f_z = 1$

The $grad(f) = f_x i + f_y j + f_z k = (2x-6)i + (2y-8)j + k$ **. The unit vector of** $i - 2j + k = \frac{1}{\sqrt{14}}(i - 2j + k)$.

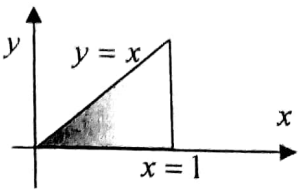
The rate of change at point (2, 2, 20) is $(-2i - 4j + k) \odot \frac{1}{\sqrt{14}}(i - 2j + k) = \frac{9}{\sqrt{14}}$

The maximum rate of change at point (2, 2, 20) is $\|grad(f)\| = \|(-2i - 4j + k)\| = \frac{4+16+1}{\sqrt{21}} = \sqrt{21}$

Question 4 (c) (i)[3 Marks]

(i) Change the order of integration, $I = \int_0^1 \int_0^x e^{x^2} dy dx = \int_0^1 (y)_0^x \, e^{x^2} \, dx = \int_0^1 (x-0) \, e^{x^2} \, dx$

$I = \frac{1}{2}\int_0^1 2x \, e^{x^2} \, dx = \frac{1}{2}\left|e^{x^2}\right|_0^1 = \frac{1}{2}(e-1)$

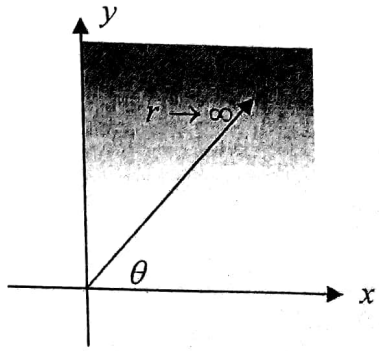


Question 4 (c) (ii)[3 Marks]

(ii) Change to polar, $I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy = \int_0^{\pi/2} \int_0^\infty e^{-r^2} \, r dr d\theta$

$I = \frac{-1}{2}\int_0^{\pi/2} \int_0^\infty (-2r) e^{-r^2} \, dr d\theta = \frac{-1}{2}\int_0^{\pi/2} \left|e^{-r^2}\right|_0^\infty d\theta$

$I = \frac{-1}{2}\left(e^{-\infty} - e^0\right)\int_0^{\pi/2} d\theta = \frac{-1}{2}(0-1)(\theta)_0^{\pi/2} = \frac{\pi}{4}$



$x = r\cos\theta, \quad y = r\sin\theta$

$x^2 + y^2 = r^2, \; dxdr = rdrd\theta$

$0 < r < \infty, \quad 0 < \theta < \pi/2$

الدرجة ليست على الناتج النهائى

Best of luck, Prof. Ibrahim L. El-Kalla

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