Mansoura University
Faculty of Engineering
BME
Course Title: Differential Equations
Course Code: Math(106)

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Final Exam
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[1]-(a) Find the extremum of the function

$$f(x,y) = \frac{x^5}{5} - x + \cos y$$ [5 pts]

(b) If $\phi(x,y,z) = x^2 y z^3$ and $A = x z\,\mathbf{i} - y^2\,\mathbf{j} + 2x^2 y\,\mathbf{k}$
find

$$\nabla\phi \qquad \nabla\cdot A \qquad \text{Curl}\,(\phi A)$$ [5 pts]

(c) Evaluate

$$\int\int_A e^{x/y}\,dy\,dx$$

where A is bounded by parabola $y^2 = x$ and the straight lines $x = 0$, $y = 1$. [5 pts]

(d) Evaluate

$$\int_0^1\int_0^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\,dy\,dx, \qquad \int_1^3\int_x^{x^2}\int_0^{\ln z} x e^y\,dy\,dz\,dx$$ [10 pts]

(e) If $z = xy + x f\left(\frac{y}{x}\right)$
Find

$$x z_x + y z_y$$ [5 pts]

[2]-(a) Find the orthogonal trajectories of

$$x^2 + 2xy - y^2 + 4x - 4y = C$$ [5 pts]

(b) Solve by any method

1. $(\sin x \cosh y)\,dx - (\cos x \sinh y)\,dy = 0$ [5 pts]
2. $x^3 y''' - 4x^2 y'' + 8xy' - 8y = 4\ln x, \quad x > 0.$ [5 pts]

(c) Find a linear homogeneous constant-coefficient equation with the given general solution.

$$y(x) = (A + Bx + Cx^2)\,e^{2x}$$ [5 pts]

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SOLUTION

[1]-(a) Find the extremum of the function

$$f(x,y) = \frac{x^5}{5} - x + \cos y$$

SOLUTION

$$f_x = 0$$

$$x^4 - 1 = 0 \quad \Rightarrow \quad x = \pm 1, \quad \text{or} \quad x = \pm i$$

$$f_y = 0$$

$$-\sin y = 0 \quad \Rightarrow \quad y = n\pi$$

for real max(min) consider only real roots.

Critical points:

$$(1, n\pi), \quad (-1, n\pi), \quad n = 0, 1, 2, \ldots$$

$$f_{xx} = 4x^3, \qquad f_{yy} = -\cos y$$

$$f_{yy} = 0, \qquad D(x,y) = f_{xx} f_{yy} - (f_{xy})^2 = -4x^3 \cos y$$

At $(1, n\pi)$

$$D(x,y)|_{(1,n\pi)} = -4\cos n\pi = -4(-1)^n \qquad \text{if}$$

- n even $\quad D = -4 \quad$ saddle point
- n odd $\quad D = 4 > 0 \quad f_{xx} = 4 > 0 \quad$ at $(1, (2n+1)\pi) \quad \Rightarrow \quad f$ is min.

At $(-1, n\pi)$

$$D(x,y)|_{(-1,n\pi)} = 4\cos n\pi = 4(-1)^n \qquad \text{if}$$

- n even $\quad D = 4 > 0 \quad f_{xx} = -4 < 0 \quad \Rightarrow \quad$ max
- n odd $\quad D = -4 < 0 \quad \Rightarrow \quad$ saddle point.

Then,

- f has points of minimum at $(1, (2n+1)\pi)$
- f has points of maximum at $(-1, 2n\pi)$.

(b) If

$$\phi(x,y,z) = x^2 y z^3 \qquad \text{and} \qquad A = x z\,\mathbf{i} - y^2\,\mathbf{j} + 2x^2 y\,\mathbf{k}$$

find

$$\nabla\phi, \qquad \nabla \cdot A, \qquad \text{Curl}\,(\phi A)$$

SOLUTION

$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

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$$\nabla \cdot A = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot \left[x\,z\,\mathbf{i} - y^2\,\mathbf{j} + 2\,x^2\,y\,\mathbf{k}\right]$$
$$= z - 2\,y$$

$$\phi\,A = x^3\,y\,z^4\,\mathbf{i} - x^2\,y^3\,z^3\,\mathbf{j} + 2\,x^4\,y^2\,z^3\,\mathbf{k}$$

$$\text{Curl}\,(\phi\,A) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3\,y\,z^4 & -x^2\,y^3\,z^3 & 2\,x^4\,y^2\,z^3 \end{vmatrix}$$
$$= \left(4\,x^4\,y\,z^3 + 3\,x^2\,y^3\,z^2\right)\,\mathbf{i} + \left(4\,x^3\,y\,z^3 - 8\,x^3\,y^2\,z^3\right)\,\mathbf{j}$$
$$- \left(2\,x\,y^3\,z^3 + x^3\,z^4\right)\,\mathbf{k}$$

(c) Evaluate

$$\int\int_A e^{x/y}\,d\,y\,d\,x$$

where A is bounded by parabola $y^2 = x$ and the straight lines $x = 0,\ y = 1$. [5 pts]

SOLUTION

$$\int\int_A e^{x/y}\,d\,y\,d\,x = \int_0^1 \int_0^{y^2} e^{x/y}\,d\,x\,d\,y$$
$$= \int_0^1 \int_0^{y^2} e^{x/y}\,d\,x\,d\,y$$
$$= \int_0^1 \left[y\,e^{x/y}\right]_0^{y^2}\,d\,y$$
$$= \int_0^1 \left[y\,e^y - 1\right]\,d\,y$$
$$= \left[y\,e^y - e^y - y\right]_{y=0}^{y=1}$$
$$= \left[e - e - 1 + 1\right] = 0.$$

(d) Evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{x^2 + y^2}\,d\,y\,d\,x. \qquad \int_1^3 \int_x^{x^2} \int_0^{\ln z} x\,e^y\,d\,y\,d\,z\,d\,x$$ [10 pts]

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$$\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\, dy\, dx = \int_0^{\pi/2} \int_0^1 r\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \int_0^1 r^2\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^3}{3}\right]_0^1 d\theta$$

$$= \frac{1}{3} \int_0^{\pi/2} d\theta$$

$$= \frac{1}{3} [\theta]_0^{\pi/2}$$

$$= \frac{\pi}{6}.$$

$$\int_1^3 \int_x^{x^2} \int_0^{\ln z} x\, e^y\, dy\, dz\, dx = \int_1^3 \int_x^{x^2} x\, [e^y]_0^{\ln z}\, dz\, dx$$

$$= \int_1^3 \int_x^{x^2} x\, [z-1]\, dz\, dx$$

$$= \int_1^3 x \left[\frac{z^2}{2} - z\right]_x^{x^2} dx$$

$$= \int_1^3 x \left[\frac{x^4}{2} - \frac{3x^2}{2} + x\right] dx$$

$$= \int_1^3 \left[\frac{x^5}{2} - \frac{3x^3}{2} + x^2\right] dx$$

$$= \left[\frac{x^6}{12} - \frac{3x^4}{8} + \frac{x^3}{3}\right]_1^3.$$

(e) If $\quad z = xy + x f\left(\frac{y}{x}\right)$

Find

$$x z_x + y z_y$$ [5 pts]

SOLUTION

$$z = u + v, \qquad u = xy \qquad v = x f\left(\frac{y}{x}\right)$$

$$z_x = u_x + v_x \qquad x z_x = x u_x + x v_x$$

$$z_y = u_y + v_y \qquad y z_y = y u_y + y v_y$$

u is homog of degree 2 $\qquad \Rightarrow x u_x + y u_y = 2u$

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$$x\,z_x + y\,z_y = x\,u_x + x\,v_x + y\,u_y + y\,v_y$$
$$= [x\,u_x + y\,u_y] + (x\,v_x + y\,v_y)$$
$$= 2u + v = 2\,x\,y + x\,f\left(\frac{y}{x}\right).$$

ANOTHER SOLUTION

$$z_x = y + f\left(\frac{y}{x}\right) + x\,f'\left(\frac{y}{x}\right)\left[\frac{-y}{x^2}\right]$$
$$x\,z_x = x\,y + x\,f\left(\frac{y}{x}\right) - y\,f'\left(\frac{y}{x}\right) \qquad (1)$$
$$z_y = x + x\,f'\left(\frac{y}{x}\right)\frac{1}{x}$$
$$y\,z_y = x\,y + y\,f'\left(\frac{y}{x}\right) \qquad (2)$$

Then

$$xz_x + yz_y = 2\,x\,y + x\,f\left(\frac{y}{x}\right).$$

[2]-(a) Find the orthogonal trajectories of

$$x^2 + 2xy - y^2 + 4x - 4y = C$$

SOLUTION

$$2x + 2xy' + 2y - 2yy' + 4 - 4y' = 0$$
$$(2x - 2y - 4)y' = -(2x + 2y + 4)$$
$$y' = -\frac{x + y + 2}{x - y - 2}$$
$$-\frac{1}{y'} = -\frac{x + y + 2}{x - y - 2}$$

or

$$(x - y - 2)dx - (x + y + 2)\,dy = 0$$
$$\frac{\partial M}{\partial y} = -1 = \frac{\partial N}{\partial x}$$
$$f = \int (x - y - 2)dx + h(y)$$
$$= \frac{x^2}{2} - xy - 2x + h(y)$$
$$\frac{\partial f}{\partial y} = -x - 2 + h'(y)$$
$$= N = -x - y - 2$$
$$h'(y) = -y$$
$$\ldots \qquad y^2$$

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$$f = \frac{x^2}{2} - xy - 2x - \frac{y}{2} + k = 0$$

or

$$\frac{x^2}{2} - xy - 2x - \frac{y^2}{2} = k.$$

(b) Solve by any method

1. $(\sin x \cosh y)\, dx - (\cos x \sinh y)\, dy = 0$ **[5 pts]**
2. $x^3 y''' - 4x^2 y'' + 8xy' - 8y = 4 \ln x. \qquad x > 0.$ **[5 pts]**

1. SOLUTION

$$(\sin x \cosh y)\, dx - (\cos x \sinh y)\, dy = 0$$

$$\frac{\sin x}{\cos x}\, dx - \frac{\sinh y}{\cosh y}\, dy = 0$$

$$\int \frac{\sin x}{\cos x}\, dx - \int \frac{\sinh y}{\cosh y}\, dy = c$$

$$-\ln \cos x - \ln \cosh y = c.$$

2. SOLUTION

Let

$$x = e^t \qquad xy' = \Theta\, y \qquad xy'' = \Theta\,(\Theta - 1)\, y \qquad xy''' = \Theta\,(\Theta - 1)\,(\Theta - 2)\, y$$

$$[\Theta(\Theta - 1)(\Theta - 2) - 4\Theta(\Theta - 1) + 8\Theta - 8] = 4t$$

$$\left[\Theta^3 - 7\,\Theta^2 + 14\,\Theta - 8\right] = 4t$$

$$y_g = y_c + y_p$$

To find y_c:

$$\left[\Theta^3 - 7\,\Theta^2 + 14\,\Theta - 8\right] = 0$$

Roots are $1, 2, 4$

$$y_c = c_1 e^t + c_2 e^{2t} + c_3 e^{4t}$$

To find y_p:

$$yp = A_1 t + A_0 \qquad y_p' = A_1 \qquad y_p'' = 0 \qquad y_p''' = 0$$

$$0 + 0 + 14A_1 - 8A_1 t - 8A_0 = 4t$$

Then

$$A_1 = -\frac{1}{2} \qquad A_0 = -\frac{7}{8}$$

Therefore

$$y_p = -\frac{1}{2}t - \frac{7}{8}.$$

(c) Find a linear homogeneous constant-coefficient equation with the given general solution.

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SOLUTION

$$(r-2)^3 = 0$$
$$(r-2)^2(r-2) = 0$$
$$\left(r^2 - 4r + 4\right)(r-2) = 0$$
$$r^3 - 4r^2 + 4r - 2r^2 + 8r - 8 = 0$$
$$r^3 - 6r^2 + 12r - 8 = 0$$

The DE is

$$y''' - 6y'' + 12y' - 8y = 0.$$

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