Fall Semester Exam.
Level 200
Wednesday 2/1/2018

Mansoura University
Faculty of Engineering

Biomedical Engineering Programs

Prof. Dr. Magdi S. El-Azab   Numerical Analysis (MTH 201)   Time allowed: 2 Hours.

Answer the following questions (Full mark 50 marks). (All computations have to be executed to four decimal places)

Question 1 [24 marks]

(a) [8 marks] By the least square technique, derive the normal equations to fit the curve $y = ax + b$ to the data $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$. Given the table of readings:

x	0.5	1	1.5	2	2.5
y	15.08	8.32	5.96	4.74	3.99

Fit the curve $y = \left(\dfrac{a+\sqrt{x}}{b\sqrt{x}}\right)^2$, with the computations of $L_2 - norm\ of$ the error to the above readings.

(b) [8 marks] By the use of two different interpolation formulas approximate $y(0.75)$ and $y(1.6)$.

(c) [8 marks]. Find the positive two roots of the algebraic equation

$$x^4 - 12x^2 + 35.3 = 0,$$

correct to five decimal places by two different methods, one method for each root. Guess the other two roots.

باقى الاسئلة فى الصفحة التالية

Question 2 [27 marks]

(a) [9 marks] Given the initial value problem:

$$y' = \sqrt{10x^2 + y}, \qquad x_0 = 0.2, \qquad y_0 = 0.41,$$

(i) Evaluate $y(0.4)$ by Runge-Kutta approximation of order 4,

(ii) Find $y(0.7)$ by the use of Taylor method,

(iii) Find $y(0.8)$ by the use of Euler method.

(b) [9 marks] Starting with the initial values

$$\left(x_1^{(0)},\ x_2^{(0)},\ x_3^{(0)},\ x_4^{(0)}\right) = (2, 2, -2, -2),$$

solve the following system of equations correct to three decimal places

$$\begin{aligned} 2x_1 + 0.5x_2 + 0.5x_3 + x_4 &= 6 \\ x_1 + 4x_2 - x_3 + x_4 &= 8 \\ 2x_1 - x_2 + 4x_3 - 0.5x_4 &= 5 \\ 2x_1 + x_2 + 3x_3 + 5x_4 &= 2 \end{aligned}$$

(c) [9 marks] Given the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 12(x + y), \quad 0 < x < 2, \ 0 < y < 2$$

Take the mesh sizes $h = k = 0.5$ and the unknown u is given to be $4(4x^2 + y)$ on the boundaries. Apply the finite difference method and the relaxation method to solve this problem with the initial values

$$\begin{aligned} u_{11}^{(0)} &= 4, & u_{21}^{(0)} &= 6, & u_{31}^{(0)} &= 8, \\ u_{12}^{(0)} &= 10, & u_{22}^{(0)} &= 12, & u_{32}^{(0)} &= 14, \\ u_{13}^{(0)} &= 16, & u_{23}^{(0)} &= 18, & u_{33}^{(0)} &= 20 \end{aligned}$$

With all best wishes

Question 1 [24 marks]

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x	0.5	1	1.5	2	2.5
y	15.08	8.32	5.96	4.74	3.99

Fit the curve $y = \left(\frac{a+\sqrt{x}}{b\sqrt{x}}\right)^2$, with the computations of $L_2 - norm\ of$ the error to the above readings.

Answer. Let $y(x)$ be a function and $x_r,\ r = 1,2,\cdots,n$ be a sequence of data points at which we have observed values of $y(x)$ which generally will be in error. We denote $y(x_r)$, the true value at x_r, and we denote the observed value at x_r by y_r. So we define the deviations at the data points by (the 'error' between data and true values) (see Fig. 1.1)

$$d_1 = y(x_1) - y_1$$
$$d_2 = y(x_2) - y_2$$
$$d_3 = y(x_3) - y_3$$
$$\vdots$$

x_r	y_r	$y(x_r)$	$d_r = y(x_r) - y_r$
x_1	y_1	$y(x_1)$	$d_1 = y(x_1) - y_1$
x_2	y_2	$y(x_2)$	$d_2 = y(x_2) - y_2$
x_3	y_3	$y(x_3)$	$d_3 = y(x_3) - y_3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

or, in general

$$d_r = y(x_r) - y_r, \quad r = 1,2,\cdots,n$$

We shall assume that the errors at different data points are independent. Thus our problem is to minimize the deviations $d_r,\ r = 1,2,\cdots,n$ in some sense. We concern with the *least square technique* at which we minimize the sum of the squares of deviations d_r. Let

$$y(x) = ax + b,$$

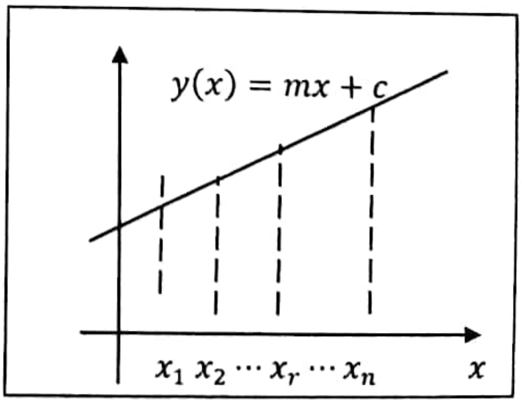
The general idea is to choose the constants a and b to minimize the error. It is clear that the minimization of a function of two variables needs to set its partial derivatives with respect to these variables to zero and simultaneously solve the resulting equations.

The least squares approach to this problem involves of minimizing the best approximating line when the error involved is the sum of the squares of the differences between the values on the approximating line and the given values ($L_2 - norm$);

$$\|d\| = \left(\sum_{r=1}^{n} {d_r}^2\right)^{1/2},$$

or equivalently we minimize

$$e(a,b) = \|d\|^2 = \sum_{r=1}^{n} {d_r}^2$$
$$= \sum_{r=1}^{n} (y(x_r) - y_r)^2$$
$$= \sum_{r=1}^{n} (ax_r + b - y_r)^2$$



If $\|d\|^2$ is considered to be a function of two variables $e(a,b)$, an elementary result from calculus of several variables implies, for a minimum to occur at (a,b), it is necessary for the minimization of $e(a,b)$ that

$$\frac{\partial}{\partial a} e(a,b) = 0, \quad \frac{\partial}{\partial b} e(a,b) = 0,$$

which simplifies to the linear system of equations

$$\sum_{r=1}^{n} 2(mx_r + c - y_r)(x_r) = 0, \quad \sum_{r=1}^{n} 2(mx_r + c - y_r)(1) = 0,$$

or

$$\boxed{\begin{aligned} a\sum_{r=1}^{n} x_r^2 + b\sum_{r=1}^{n} x_r &= \sum_{r=1}^{n} x_r y_r, \\ a\sum_{r=1}^{n} x_r + nb &= \sum_{r=1}^{n} y_r \end{aligned}}$$

To fit the given curve to the given readings, we write it as

$$y = \left(\frac{a+\sqrt{x}}{b\sqrt{x}}\right)^2 \Rightarrow \sqrt{y} = \frac{a+\sqrt{x}}{b\sqrt{x}} \Rightarrow \sqrt{y} = \frac{a}{b\sqrt{x}} + \frac{1}{b}$$

$Y = AX + B$	$X = \frac{1}{\sqrt{x}}, \quad Y = \sqrt{y}$
	$A = \frac{a}{b} \Rightarrow a = bA, \quad B = \frac{1}{b} \Rightarrow b = \frac{1}{B}$

So that the normal equations shall be

$$\begin{aligned} A\sum_{r=1}^{n} X_r^2 + B\sum_{r=1}^{n} X_r &= \sum_{r=1}^{n} X_r Y_r, \\ A\sum_{r=1}^{n} X_r + 5\,B &= \sum_{r=1}^{n} Y_r \end{aligned}$$

Construct the table

x	y	$X = \frac{1}{\sqrt{x}}$	$Y = \sqrt{y}$	X^2	XY	y approx	d^2
0.5	15.08	1.414214	3.883298	2	5.491812	15.08104	1.08E-06
1	8.32	1	2.884441	1	2.884441	8.318033	3.87E-06
1.5	5.96	0.816497	2.441311	0.666667	1.993322	5.960335	1.12E-07
2	4.74	0.707107	2.177154	0.5	1.53948	4.741357	1.84E-06
2.5	3.99	0.632456	1.997498	0.4	1.263329	3.989456	2.96E-07
Σ		4.570272	13.3837	4.566667	13.17238		7.2E-06

Substitute in the normal equations we obtain

$$\begin{aligned} 4.566667A + 4.570272\,B &= 13.17238, \\ 4.570272A + 5\,B &= 13.3837 \end{aligned}$$

Solving this system, we get

$$\begin{aligned} A &= 2.4126 \quad \Rightarrow \quad a = 5.1168611 \\ B &= 0.4715 \quad \Rightarrow \quad b = 2.120891 \\ L_2 - \textit{norm of the error} &= 0.002684 \end{aligned}$$

From which we have

$$\boxed{y = \left(\frac{5.1168611 + \sqrt{x}}{2.120891\sqrt{x}}\right)^2}$$

(b) [8 marks] By the use of two different interpolation formulas approximate $y(0.75)$ and $y(1.6)$.

Answer. Using Lagrang's interpolation formula

x	0.5	1	1.5	2	2.5
y	15.08	8.32	5.96	4.74	3.99
$l =$	0.273	1.09375	-0.54688	0.21875	-0.03906
	0.014	-0.1056	0.9504	0.1584	-0.0176

$$y(0.75) = 10.84507813, \qquad y(1.6) = 5.683536$$

x	y	Δ	Δ^2	Δ^3	Δ^4
0.5	15.08				
		-6.76			
1	8.32		4.4		
		-2.36		-3.26	
1.5	5.96		1.14		2.59
		-1.22		-0.67	
2	4.74		0.47		
		-0.75			
2.5	3.99				

f(	2	) =	3.3
f(	2.5	) =	-0.6375
f(	3	) =	8.3
f(	3.5	) =	38.3625

Using Newton forward interpolation formulas to get $y(0.75)$ we get

$$x_0 = 0.5, \quad x = 0.75 \quad \Rightarrow \quad h = 0.5, \quad s = 0.5$$

$$y(0.75) = 15.08 + (-6.76)(0.5) + (4.4)\left(\frac{0.5(-0.5)}{2}\right) + (-3.26)\left(\frac{0.5(-0.5)(-1.5)}{6}\right)$$
$$+(2.59)\left(\frac{0.5(-0.5)(-1.5)(-2.5)}{24}\right) = 10.84507813$$

Using Gauss forward interpolation formulas to get $y(1.6)$we get

$$x_0 = 1.5, \quad x = 1.6 \quad \Rightarrow \quad h = 0.5, \quad s = 0.2$$

$$y(1.6) = 5.96 + (-1.22)(0.2) + (1.14)\left(\frac{0.2(-0.8)}{2}\right) + (-0.67)\left(\frac{0.2(-0.8)(1.2)}{6}\right)$$
$$+(2.59)\left(\frac{0.2(-0.8)(1.2)(-1.8)}{24}\right) = 5.621376$$

(c) [8 marks]. Find the positive two roots of the algebraic equation $x^4 - 12x^2 + 35.3 = 0$, correct to five decimal places by two different methods, one method for each root. Guess the other two roots.

Answer.

$$f(x) = x^5 - 3x^4 + 5x - 10$$

x	1	1.5	2	2.5	3
$f(x)$	24.3	13.3625	3.3	-0.6375	8.3

There is at least two root in the intervals $(2, \ 2.5)$ and $(2.5, \ 3)$ So that the positive two roots of the equation are

$$x_1 = 2.272298, \quad x_1 = 2.614701,$$

and the other two roots

$$x_3 = -2.272298, \quad x_4 = -2.614701$$

Newton-Raphson method							
n	x	$f(x)$	$f'(x)$	n	x	$f(x)$	$f'(x)$
1	2.3	-0.195900	-6.532000	1	3	8.300000	36.000000
2	2.270009	0.017509	-7.691327	2	2.769444	2.088299	18.497893
3	2.272285	0.000102	-7.605073	3	2.656550	0.417794	11.234634
4	2.272298	0.000003	-7.604580	4	2.619362	0.041420	9.021683
5	2.272298	0.000003	-7.604580	5	2.614771	0.000615	8.754540
6	2.272298	0.000003	-7.604580	6	2.614701	0.000002	8.750477
7	2.272298	0.000003	-7.604580	7	2.614701	0.000002	8.750477

Bisection method									
n	a	b	x=(a+b)/2	f(x)	a	b	x=(a+b)/2	f(x)
1	2	2.5	2.25	0.178906	2.5	2.7	2.6	-0.1224
2	2.250000	2.500000	2.375000	-0.570850	2.600000	2.700000	2.650000	0.345506
3	2.250000	2.375000	2.312500	-0.274448	2.600000	2.650000	2.625000	0.093213
4	2.250000	2.312500	2.281250	-0.066546	2.600000	2.625000	2.612500	-0.01912
5	2.250000	2.281250	2.265625	0.051591	2.612500	2.625000	2.618750	0.03591
6	2.265625	2.281250	2.273438	-0.008642	2.612500	2.618750	2.615625	0.008112
7	2.265625	2.273438	2.269532	0.021182	2.612500	2.615625	2.614063	-0.00557
8	2.269532	2.273438	2.271485	0.006198	2.614063	2.615625	2.614844	0.001254
9	2.271485	2.273438	2.272462	-0.001244	2.614063	2.614844	2.614454	-0.00216
10	2.271485	2.272462	2.271974	0.002469	2.614454	2.614844	2.614649	-0.00045
11	2.271974	2.272462	2.272218	0.000611	2.614649	2.614844	2.614747	0.000405
12	2.272218	2.272462	2.272340	-0.000316	2.614649	2.614747	2.614698	-0.00002
13	2.272218	2.272340	2.272279	0.000147	2.614698	2.614747	2.614723	0.000195
14					2.614698	2.614723	2.614711	0.000090
15					2.614698	2.614711	2.614705	0.000037

Question 2 [27 marks]

(a) [9 marks] Given the initial value problem:

$$y' = \sqrt{10x^2 + y}, \quad x_0 = 0.2, \quad y_0 = 0.41,$$

(i) Evaluate $y(0.4)$ by Runge-Kutta approximation of order 4,

(ii) Find $y(0.7)$ by the use of Taylor method,

(iii) Find $y(0.8)$ by the use of Euler method.

Answer.

$$y' = f(x, y) = \sqrt{10x^2 + y}$$

(i) RK4 method $y(0.4)$: $x_0 = 0.2, \ y_0 = 0.41 \Rightarrow h = 0.2$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + 2k_4)$$

x =	0.2	0.3	0.4
y =	0.41	0.5	0.649025
		0.528322	

$$k_1 = hf(0.5,1.5) = 0.18,$$
$$k_2 = hf(0.6,1.53465736) = 0.236643,$$
$$k_3 = hf(0.6,1.53791531) = 0.239025,$$
$$k_4 = hf(0.7,1.57598312) = 0.299935,$$

$$\boxed{y(0.4) = 1.5 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + 2k_4) = 0.648545}$$

(ii) Taylor method $y(0.7)$: $\quad x_0 = 0.4, \quad y_0 = 0.648545 \Rightarrow h = 0.3$

$y' = \sqrt{10x^2 + y}$	$y'_0 = 1.499515$
$y'' = \dfrac{20x + y'}{2\sqrt{10x^2 + y}} = \dfrac{20x + y'}{2y'} = \dfrac{10x}{y'} + \dfrac{1}{2}$	$y''_0 = 3.167529$
$y''' = \dfrac{10}{y'} - \dfrac{10xy''}{(y')^2}$	$y'''_0 = 1.034017$

$$\boxed{y(0.4) = y_1 = y_0 + hy'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \cdots = 1.245592}$$

(iii) Euler method $y(0.8)$: $\quad x_0 = 0.7, \quad y_0 = 1.245592 \Rightarrow h = 0.1$

$$\boxed{y(0.8) = y_1 = y_0 + hf(x_0, y_0) = 1.493495}$$

(b) [9 marks] Starting with the initial values $\left(x_1^{(0)},\ x_2^{(0)},\ x_3^{(0)},\ x_4^{(0)}\right) = (2, 2, -2, -2)$, solve the following system of equations correct to three decimal places

$$\begin{aligned} 2x_1 + 0.5x_2 + 0.5x_3 + \quad x_4 &= 6 \\ x_1 + \quad 4x_2 - \quad x_3 + \quad x_4 &= 8 \\ 2x_1 - \quad x_2 + \quad 4x_3 - 0.5x_4 &= 5 \\ 2x_1 + \quad x_2 + \quad 3x_3 + \quad 5x_4 &= 2 \end{aligned}$$

Answer.

	x1	x2	x3	x4	1
x1 =	0	-0.25	-0.25	-0.5	3
x2 =	-0.25	0	0.25	-0.25	2
x3 =	-0.5	0.25	0	0.125	1.25
x4 =	0	-0.25	-0.25	-0.5	3

	Gauss-Seidel method			
$k = 0$	**2**	**2**	**-2**	**-2**
$k = 1$	4.0000	1.0000	-0.7500	-0.9500
$k = 2$	3.4125	1.1969	-0.2758	-1.0389
$k = 3$	3.2892	1.3685	-0.1823	-1.0800
$k = 4$	3.2435	1.4136	-0.1534	-1.0881
$k = 5$	3.2290	1.4264	-0.1439	-1.0905
$k = 6$	3.2246	1.4305	-0.1410	-1.0913
$k = 7$	3.2233	1.4318	-0.1401	-1.0916
$k = 8$	3.2229	1.4322	-0.1399	-1.0917
$k = 9$	3.2228	1.4323	-0.1398	-1.0917
$k = 10$	3.2227	1.4323	-0.1397	-1.0917
$k = 11$	3.2227	1.4323	-0.1397	-1.0917

(c) [9 marks] Given the partial differential equation

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Take the mesh sizes $h = k = 0.5$ and the unknown u is given to be $4(4x^2 + y)$ on the boundaries. Apply the finite difference method and the relaxation method to solve this problem with the initial values

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$$u_{12}^{(0)} = 10, \quad u_{22}^{(0)} = 12, \quad u_{32}^{(0)} = 14,$$
$$u_{13}^{(0)} = 16, \quad u_{23}^{(0)} = 18, \quad u_{33}^{(0)} = 20$$

Answer.

8	12		24		44		72
6	u13	R13	u23	R23	u33	R33	70
	16	-24	18	--8	20	57	
	-6	24		18		16	
		0		10	18	73	
				-6		-72	
	10		21	4	41	1	
4	u12	R12	u22	R22	u32	R32	68
	10	-9	12	-9	14	45	
		-6		16		21	
		-15		7	16	66	
		6		8		-64	
		-9		15		2	
				8		18	
			6	23		20	
				-24		6	
				-1		26	
						6	
						32	
	10		20		38	26	
2	u11	R11	u21	R21	u31	R31	66
	4	3	6	12	8	84	
		8		21	21	-84	
		11	8	32		0	
				-32		16	
				0		16	
				6		8	
				6	6	24	
						-24	
						0	
	7		15		36		
0	4		16		36		64