Mansoura University
Faculty of Engineering

Fall Semester Exam.
Level 200
Sunday 24/1/2017

Biomedical Engineering Programs

Prof. Dr. Magdi S. El-Azab | Numerical Analysis (MTH 201) | Time allowed: 2 Hours.

Answer the following questions (Full mark 50 marks). (All computations have to be executed to four decimal places)

Question 1 [24 marks]

(a) [8 marks] Given the table of readings:

x	0.25	0.5	0.75	1	1.25
y	1.14	1.82	2.03	2.00	1.87

(i) Fit the curve $y = \dfrac{x}{a + bx^2}$, with the computations of $L_2 - norm$ of the error, to the above readings.

(ii) By the use of two different interpolation formulas approximate $y(0.4)$ and $y(0.8)$.

(b) [8 marks] Given the initial boundary-value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 2, \qquad t > 0$$

$$u(0,t) = 20, \quad u(2,t) = 0, \qquad t > 0$$

$$u(x,0) = 20\cos\left(\frac{\pi x}{4}\right), \qquad 0 < x < 2, \qquad t > 0$$

Taking $h = 0.4$, approximate $u(x, 0.16)$, $0 \le x \le 2$ correct to three decimal places.

(c) [8 marks] Derive Newton-Raphson formula to solve the equation $f(x) = 0$. Hence use this method to solve numerically for x the equation

$$x^4 - 6.255x^3 + 10.78x^2 - 6.255x + 9.78 = 0,$$

correct to five decimal places.

باقى الاسئلة فى الصفحة التالية

Question 2 [27 marks]

(a) [9 marks] Given the initial value problem:

$$y' + \frac{2x + y^2}{xy} = 0, \quad y(1.2) = 1,$$

(i) Evaluate $y(1.4)$ by Runge-Kutta approximation of order 4,

(ii) Find $y(1)$ by the use of Taylor method,

(iii) Find $y(1.6)$ by the use of Euler method.

(iv) Use a predictor-corrector method to evaluate $y(1.8)$.

(b) [9 marks] Starting with the initial values

$$\left(x_1^{(0)},\ x_2^{(0)},\ x_3^{(0)},\ x_4^{(0)}\right) = (2,\ 2,\ 5,\ 5),$$

solve the following system of equations correct to three decimal places

$$\begin{array}{rcrcrcrcr}
10x_1 & + & x_2 & - & 2x_3 & + & x_4 & = & 10 \\
x_1 & - & 10x_2 & - & x_3 & + & x_4 & = & -20 \\
2x_1 & - & 2x_2 & + & 10x_3 & - & 4x_4 & = & 50 \\
x_1 & - & x_2 & + & x_3 & - & 10x_4 & = & -40
\end{array}$$

(c) [9 marks] Apply the finite difference method and two iterations of Gauss-Seidel method to solve Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 12(x + y),$$

at the interior points of the square $0 \leq x \leq 2,\ 0 \leq y \leq 1.5$. Take the mesh sizes $h = k = 0.5$ and the unknown u is given to be $(4x^2 + 2y)$ on the boundaries.

With all best wishes

Model Answer

Question 1 [24 marks]

(a) [8 marks] Given the table of readings:

x	0.25	0.5	0.75	1	1.25
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(i) Fit the curve $y = \dfrac{x}{a + bx^2}$, with the computations of $L_2 - norm$ of the error, to the above readings.

(ii) By the use of two different interpolation formulas approximate $y(0.4)$ and $y(0.8)$.

Answer. To fit the given curve to the given readings, we write it as

$$\frac{1}{y} = \frac{a + bx^2}{x} \Rightarrow \frac{x}{y} = a + bx^2$$

$Y = A + BX$	$X = x^2, \quad Y = \dfrac{x}{y}$
	$A = a, \quad B = b$

So that the normal equations shall be

$$A\sum_{r=1}^{n} X_r + B\sum_{r=1}^{n} X_r^2 = \sum_{r=1}^{n} X_r Y_r,$$

$$5A \quad + B\sum_{r=1}^{n} X_r = \sum_{r=1}^{n} Y_r$$

Construct the table

	x	y	$X = x^2$	$Y = \dfrac{x}{y}$	X^2	XY	y approx	d^2
	0.25	1.14	0.0625	0.2193	0.00391	0.01371	1.1409	8.1E-07
	0.5	1.82	0.25	0.27473	0.0625	0.06868	1.8162	1.4E-05
	0.75	2.03	0.5625	0.36946	0.31641	0.20782	2.03293	8.6E-06
	1	2.00	1	0.5	1	0.5	2	0
	1.25	1.87	1.5625	0.66845	2.44141	1.04445	1.86979	4.5E-08
Σ			3.4375	2.03193	3.82422	1.83466		2.4E-05

Substitute in the normal equations we obtain

$$3.4375A + 3.82422\mathrm{B} = 1.83466,$$
$$5\mathrm{A} + 3.4375b = 2.03193$$

Solving this system, we get

$A =$	0.2004	$a =$	0.2004
$B =$	0.2996	$b =$	0.2996
		$Error =$	0.004889

From which we have

$$\boxed{y = \frac{x}{0.2004 + 0.2996x^2}}$$

x	y	Δ	Δ^2	Δ^3	Δ^4
0.25	1.14				
		0.68			
0.5	1.82		-0.47		
		0.21		0.23	
0.75	2.03		-0.24		-0.09
		-0.03		0.14	
1	2.00		-0.1		
		-0.13			
1.25	1.87				

Newton forward formula

x0 =	0.25	x =	0.4	h =	0.25	s =	0.6
	y (	0.4	) =				
		three terms		1.6044			
		four terms		1.61728			
		five terms		1.620304			

Lagrang's interpolation formula (five points)

x	0.25	0.5	0.75	1	1.25
y	1.14	1.82	2.03	2.00	1.87
l =	0.0144	-0.1056	0.9504	0.1584	-0.0176
	0.1904	1.1424	-0.4896	0.1904	-0.0336

y (	0.4	)	=	1.620304
y (	0.8	)	=	2.037424

(b) [8 marks] Given the initial boundary-value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 2, \qquad t > 0$$

$$u(0,t) = 20, \quad u(2,t) = 0, \qquad t > 0$$

$$u(x,0) = 20\cos\left(\frac{\pi x}{4}\right), \qquad 0 < x < 2, \qquad t > 0$$

Taking $h = 0.4$, approximate $u(x, 0.16)$, $0 \le x \le 2$ correct to three decimal places.

Answer: Choosing λ to equal 0.5. Here we have $c^2 = 1$, so that

$$\lambda = \frac{kc^2}{h^2} = \frac{k * 1}{(0.4)^2} = \frac{1}{2} \Rightarrow \quad k = 0.08,$$

Using the mesh steps h and k to create the grid, we see a portion of it in the following figure.

20	$u_{12} =$ 17.6942	$u_{22} =$ 14.635225	$u_{32} =$ 10.633125	$u_{42} =$ 5.59015	0
20	$u_{11} =$ 18.09015	$u_{21} =$ 15.3884	$u_{31} =$ 11.1803	$u_{41} =$ 5.87785	0
20	19.0211	16.1803	11.7557	6.1803	0

(c) [8 marks] Derive Newton-Raphson formula to solve the equation $f(x) = 0$. Hence use this method to solve numerically for x the equation

$$x^4 - 6.255x^3 + 10.78x^2 - 6.255x + 9.78 = 0,$$

correct to five decimal places.

Answer:

$$f(x) = x^4 - 6.255x^3 + 10.78x^2 - 6.255x + 9.78.$$

x	0	1	2	3	3.1	3.2
$f(x)$	9.78	9.05	6.35	0.15	-0.0053	0.04496

There is at least one root in the interval $(3,\ 3.1)$, $(3.1,\ 3.2)$. So

n	Newton-Raphson method	Newton-Raphson method
	x	x
0	3	4
1	3.06097561	3.638550846
2	3.085135537	3.413502336
3	3.091514857	3.28264007
4	3.092052477	3.212013772
5	3.092056383	3.177864174
6	3.092056383	3.165245596
7	3.092056383	3.16301667
8	3.092056383	3.162943695
9	3.092056383	3.162943617

Question 2 [27 marks]

(a) [9 marks] Given the initial value problem:

$$y' + \frac{2x + y^2}{xy} = 0, \quad y(1.2) = 1,$$

(i) Evaluate $y(1.4)$ by Runge-Kutta approximation of order 4,

(ii) Find $y(1)$ by the use of Taylor method,

(iii) Find $y(1.6)$ by the use of Euler method.

(iv) Use a predictor-corrector method to evaluate $y(1.8)$.

Answer:

$$y' = f(x) = -\frac{2x + y^2}{xy}, \quad y(1.2) = 1$$

(i) RK4 - method y at x = 1.4 h = 0.2

x =	1.2	1.3	1.4
y =	1	0.716667	0.581215
		0.828352	

k1 = 0.56667 k2 = -0.3433 k4 = -0.26053

k3 = -0.41879

$$y(1.4) = y(1) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + 2k_4) = 0.608106$$

(ii) Taylor method to get $y(1)$. Take $x_0 = 1.2, \quad y_0 = 1 \Rightarrow h = -0.2$

y' =	- (2x+y^2) / x y = -(2/y)-(y/x))	y' =	-2.83333
y'' =	(2y'/y^2)+(y/x^2)-(y'/x)	y'' =	-2.72685
y''' =	(2y''/y^2)-(4y'^2/y^3)+(2y'/x^2)-(2y/x^3)-(y''/x)	y''' =	-40.385

$$y(1) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots = 1.584155$$

(ii) Euler method to get $y(1.6)$. Take $x_0 = 1.4, \quad y_0 = 0.608106 \Rightarrow h = 0.2$

$$y(1.6) = y_0 + hy_0' = 1.000712$$

(iii) Predictor-corrector methods to get $y(1.8)$.

	1	-5	19	9
-9	37	-59	55	
1.58416	1	0.60811	1.00071	1.5505
1	1.2	1.4	1.6	1.8
2.25863	2.09545	1.96303	2.26527	2.58683

y1* = 1.550501938 and y1c = 1.489060613

$$y^p(1.8) = y_0 + \frac{h}{24}(55f_3 - 59f_2 + 37f_1 - 9f_0) = 0.9685,$$

$$y^c(1.8) = y_0 + \frac{h}{24}(9f_4 + 19f_3 - 5f_2 + f_0) = 0.5215$$

(b) [9 marks] Starting with the initial values

$$\left(x_1^{(0)},\ x_2^{(0)},\ x_3^{(0)},\ x_4^{(0)}\right) = (2,\ 2,\ 5,\ 5),$$

solve the following system of equations correct to three decimal places

$$\begin{array}{rcrcrcrcr} 10x_1 & + & x_2 & - & 2x_3 & + & x_4 & = & 10 \\ x_1 & - & 10x_2 & - & x_3 & + & x_4 & = & -20 \\ 2x_1 & - & 2x_2 & + & 10x_3 & - & 4x_4 & = & 50 \\ x_1 & - & x_2 & + & x_3 & - & 10x_4 & = & -40 \end{array}$$

Answer: Solving three equations in three unknowns

	x_1	x_2	x_3	x_4	1
$x_1 =$	0	-0.1	0.2	-0.1	1
$x_2 =$	0.1	0	-0.1	0.1	2
$x_3 =$	-0.2	0.2	0	0.4	5
$x_4 =$	0.1	-0.1	0.1	0	4

		Gauss-Seidel method			
$k =$	0	2	2	5	5
$k =$	1	1.3000	2.1300	7.1660	4.6336
$k =$	2	1.7568	1.9224	6.8866	4.6721
$k =$	3	1.7179	1.9503	6.9153	4.6683
$k =$	4	1.7212	1.9474	6.9126	4.6686
$k =$	5	1.7209	1.9477	6.9128	4.6686
$k =$	6	1.7209	1.9477	6.9128	4.6686

	x_1	x_2	x_3	x_4	1
$x_1 =$	0	-0.1	0.2	-0.1	1
$x_2 =$	0.1	0	-0.1	0.1	2
$x_3 =$	-0.2	0.2	0	0.4	5
$x_4 =$	0.1	-0.1	0.1	0	4

		Gauss-Jacobi method			
$k =$	0	2	2	5	5
$k =$	1	1.3000	2.2000	7.0000	4.5000
$k =$	2	1.7300	1.8800	6.9800	4.6100
$k =$	3	1.7470	1.9360	6.8740	4.6830
$k =$	4	1.7129	1.9556	6.9110	4.6685
$k =$	5	1.7198	1.9470	6.9159	4.6668
$k =$	6	1.7218	1.9471	6.9122	4.6689
$k =$	7	1.7208	1.9479	6.9126	4.6687
$k =$	8	1.7209	1.9477	6.9129	4.6686

(c) [9 marks] Apply the finite difference method and two iterations of Gauss-Seidel method to solve Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 12(x + y),$$

at the interior points of the square $0 \leq x \leq 2,\ 0 \leq y \leq 1.5$. Take the mesh sizes $h = k = 0.5$ and the unknown u is given to be $(4x^2 + 2y)$ on the boundaries.

Answer: The values of u at the boundary points are calculated from the equation $u = (4x^2 + 2y)$ and taking the starting values,

$$u_{11} = 1, \quad u_{21} = 3, \quad u_{31} = 5,$$
$$u_{12} = 2, \quad u_{22} = 4, \quad u_{32} = 6,$$

we get