Mansoura University Faculty of Engineering Dept. of Power Mech. Eng. Course Title: 3D Modeling & Simulation of Heat Transfer & Fluid Flow Course Code: MPE371		BME Level 300 January 2019 Exam Type: Final Time: 2 Hours Full Mark: 50

Attempt All Five Questions. Assume Any Missing Data. Questions Are Equal Weight But Are Not Equal Difficulty.

Question (1)

1.a Mention the names of three conservation laws.

1.b Write an example for ordinary differential equation (ODE) in fluid mechanics.

1.c Write the Fourier's law of heat conduction. Why do we put the negative sign in the Fourier's law of heat conduction?. When does the heat flux (q) equal zero?

1.d For one-dimensional transient (unsteady) heat conduction, what is the type of this partial differential equation (PDE)? Why? What are the name & units of property (α) in this equation?

Question (2)

2.a An incompressible steady-flow pattern is given by $u = x^3 + 2z^2$ and $w = y^3 - 2yz$. What is the most general form of the third component, v(x, y, z), which satisfies continuity?

2.b Mention the names of different forces in the momentum equation.

2.c. When does the gravitational force has a value in the momentum equation?. When is this value positive?. When is this value negative?.

Question (3)

For the following velocity distribution.

$$u/U_\infty = a + b\eta + c\eta^2 + d\eta^3$$

Where a, b, c, d are constants and $\eta = y/\delta$

i. Write the four boundary conditions required to determine the constants a, b, c, d.

ii. By solving these four equations, obtain the values of constants a, b, c, d.

iii. Calculate the displacement thickness (δ^*) and momentum thickness (θ) in terms of δ.

Question (4)

4.a Consider the steady-state two-dimensional heat conduction. Show that the steady-state temperature distribution on a heated plate can be modeled using the equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

For the square grid ($\Delta x = \Delta y$) and using the approximation of $\partial^2 T/\partial x^2$ and $\partial^2 T/\partial y^2$, show that the above equation can be written as

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$$

4.b What is the difference between Laplace equation & Poisson equation? What is the type of these partial differential equations (PDEs)? Why?

Question (5)

Consider the problem of heat conduction with uniform heat generation (q) in a rod whose ends are maintained at constant temperatures of $T_0 = 100°C$ and $T_5 = 200°C$ respectively. The one-dimensional problem is governed

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)+q=0$$

Take rod length L equals 2 cm, thermal conductivity k equals 0.5 W/m.K, heat generation q equals 10^6 W/m^3, cross-sectional area A is 1 m^2, $\delta x = 0.5$ cm.

i. Integrate the above equation 2 times with respect to x. Use the 2 boundary conditions at rod ends to obtain the values of the constants C_1 & C_2. Use this temperature distribution equation to obtain the values of the temperature T_1, T_2, T_3, T_4 at x = 2.5, 7.5, 12.5, 17.5 mm respectively.

ii. Using the control volume method, obtain the discretised equation for nodal points 2 and 3, the discretised equation for nodal point 1, and the discretised equation for nodal point 4.

Good Luck

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model ans.

Mansoura University Faculty of Engineering Biomedical Engineering	3D Modeling January 2019	Final Exam Level 300 Time allowed 2 Hrs

Question 1:

1.a The names of three conservation laws:

i. Conservation of mass.

ii. Conservation of momentum.

iii. Conservation of energy.

1.b An example for ordinary differential equation (ODE) in fluid mechanics is the law of shear stress (τ)

$$\tau = -\mu * dU/dy$$

1.c The Fourier's law of heat conduction is

$$q = -k * dT/dx$$

We put the negative sign in the Fourier's law of heat conduction because heat moves "downhill" from high to low temperature, the flow is from left to right in the positive x direction. However, due to the orientation of Cartesian coordinates, the slope (dT/dx) is negative for this case. Thus, a negative gradient (dT/dx) leads to a positive flow (q). This is the origin of the minus sign in Fourier's law of heat conduction.

The heat flux (q) equals zero when the temperature is maximum (i.e. the temperature gradient (dT/dx) is zero).

or when there is no temperature difference across the body ($\Delta T = 0$).

or when the body surface is insulated.

1.d The type of partial differential equation (PDE) for one-dimensional transient (unsteady) heat conduction

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

Or

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

For $Au_{xx}+Bu_{tx}+C\,u_{tt}+D = 0$

We have B = C = 0

B^2-4AC = 0, PDE is Parabolic

The name of property (α) in this equation is thermal diffusivity.

the units of property (α) in this equation is m^2/s.

Question 2:

2.a For an idealized incompressible flow has the proposed three-dimensional velocity distribution, the continuity relation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (1)$$

For the proposed three-dimensional velocity distribution V = ui + vj + wk, we have

$$u = x^3 + 2z^2 \text{ and } w = y^3 - 2yz$$

Therefore, $\partial u/\partial x = 3x^2$, $\partial w/\partial z = -2y$. Substitute these derivatives in Eq. (1), we have

$$3x^2 + (\partial v/\partial y) - 2y = 0$$

$$\partial v/\partial y = -3x^2 + 2y$$
$$v = \int (-3x^2 + 2y)\, dy = -3x^2y + y^2 + \text{constant}$$

2.b The names of different forces in the momentum equation:
i. Shear stress force.
ii. Pressure force.
iii. Body force.
2.c. The gravitational force has a value in the momentum equation when the pipe is vertical or inclined.
This value is positive when the flow is downwards.
This value is negative is upwards.

Question 3:

$$u/U_\infty = a + b\eta + c\eta^2 + d\eta^3$$

i. Boundary Condition 1, at $y = 0$ ($\eta = 0$), $u/U_\infty = 0$ (i)
Boundary Condition 2, at $y = \delta$ ($\eta = 1$), $u/U_\infty = 1$ (ii)
Boundary Condition 3, at $y = \delta$ ($\eta = 1$), $d(u/U_\infty)/d\eta = 0$ (iii)
Boundary Condition 4, at $y = 0$ ($\eta = 0$), $d^2(u/U_\infty)/d\eta^2 = 0$ (iv)
ii. From Boundary Condition 1, $a = 0$
ii. From Boundary Condition 2, $b + c + d = 1$
ii. From Boundary Condition 3, $b + 2c + 3d = 0$
ii. From Boundary Condition 4, $2c = 0$
Therefore, we have, $a = 0$, $b = 1.5$, $c = 0$, $d = -0.5$

$$u/U_\infty = 1.5\eta - 0.5\,\eta^3$$

iii. The displacement thickness $(\delta^*) = \delta \int (1 - u/U_\infty)\, d\eta$ for η from 0 to 1
$\delta^* = \delta \int (1 - 1.5\eta + 0.5\,\eta^3)\, d\eta$ for η from 0 to 1
$\delta^* = \delta * (\eta - 0.75\eta^2 + 0.125\,\eta^4)$ for η from 0 to 1
$\delta^* = 0.375\delta = 3\delta/8 = 105\delta/280$
iii. The momentum thickness $(\theta) = \delta \int u/U_\infty * (1 - u/U_\infty)\, d\eta$ for η from 0 to 1
$\theta = \delta \int (1.5\eta - 0.5\,\eta^3) * (1 - 1.5\eta + 0.5\,\eta^3)\, d\eta$ for η from 0 to 1
$\theta = \delta \int (1.5\eta - 2.25\,\eta^2 + 0.75\,\eta^4 - 0.5\,\eta^3 + 0.75\,\eta^4 - 0.25\,\eta^6)\, d\eta$ for η from 0 to 1
$\theta = \delta \int (1.5\eta - 2.25\,\eta^2 - 0.5\,\eta^3 + 1.5\,\eta^4 - 0.25\,\eta^6)\, d\eta$ for η from 0 to 1
$\theta = \delta * (0.75\,\eta^2 - 0.75\,\eta^3 - 0.125\,\eta^4 + 0.3\,\eta^5 - (0.25/7)\,\eta^7)$ for η from 0 to 1
$\theta = \delta * (-0.125 + 0.3 - (0.25/7)) = \delta * (-(35/280) + (84/280) - (10/280)) = 39\delta/280$

Question 4:
4.a For the steady-state two-dimensional heat conduction, the flow of heat into the element over a unit time period Δt must equal the flow out, as in

$$q(x)\,\Delta y\,\Delta z\,\Delta t + q(y)\,\Delta x\,\Delta z\,\Delta t = q(x+\Delta x)\,\Delta y\,\Delta z\,\Delta t + q(y+\Delta y)\,\Delta x\,\Delta z\,\Delta t$$

where $q(x)$ and $q(y)$ = the heat fluxes at x and y, respectively [W/m^2]. Dividing by Δz and Δt and collecting terms yields

$$[q(x) - q(x+\Delta x)]\Delta y + [q(y) - q(y+\Delta y)]\Delta x = 0$$

Multiplying the first term by $\Delta x/\Delta x$ and the second by $\Delta y/\Delta y$ gives

$[q(x) - q(x+\Delta x)]\Delta x\Delta y/\Delta x + [q(y) - q(y+\Delta y)]\Delta x\Delta y/\Delta y = 0$

Dividing by $\Delta x\Delta y$

$[q(x) - q(x+\Delta x)]/\Delta x + [q(y) - q(y+\Delta y)]/\Delta y = 0$

Taking the limit results in

$$-\frac{\partial q}{\partial x}-\frac{\partial q}{\partial y}=0$$

The above equation can be reformulated in terms of temperature. The link between flux and temperature is provided by Fourier's law of heat conduction, which can be represented as

$$q_x = -k*\partial T/\partial x$$

$$q_y = -k*\partial T/\partial y$$

Thus

$$k\frac{\partial^2 T}{\partial x^2}+k\frac{\partial^2 T}{\partial y^2}=0$$

Or

$$k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)=0$$

Since the thermal conductivity (k) is not equal to zero ($k \neq 0$), we have

The steady-state temperature distribution on a heated plate can be modeled using the equation

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0$$

Substituting the approximation of $\partial^2T/\partial x^2$,

$$\frac{\partial^2 T}{\partial x^2}\cong\frac{T_{i+1,j}-2T_{i,j}+T_{i-1,j}}{\Delta x^2}$$

and the approximation of $\partial^2T/\partial y^2$,

$$\frac{\partial^2 T}{\partial y^2}\cong\frac{T_{i,j+1}-2T_{i,j}+T_{i,j-1}}{\Delta y^2}$$

into the steady-state temperature distribution on a heated plate, we have

$$\frac{T_{i+1,j}-2T_{i,j}+T_{i-1,j}}{\Delta x^2}+\frac{T_{i,j+1}-2T_{i,j}+T_{i,j-1}}{\Delta y^2}=0$$

For the square grid ($\Delta x = \Delta y$), we have

$$T_{i+1,j}+T_{i-1,j}+T_{i,j+1}+T_{i,j-1}-4T_{i,j}=0$$

4.b The type of partial differential equation (PDE)
For $Au_{xx}+Bu_{yx}+C\, u_{yy}+D = 0$
We have A = C = 1, B = 0
B^2-4AC = -4 < 0, PDE is Elliptic

Question 5:

i.
$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)+q=0$$

The analytical solution to this problem may be obtained by integrating equation twice with respect to x and by subsequent application of the boundary conditions.
Since the thermal conductivity (k) is constant, we have

$$k\frac{d^2T}{dx^2}+q=0$$

Dividing both sides by the thermal conductivity (k), we have

$$\frac{d^2T}{dx^2}+\frac{q}{k}=0$$

Integrate the above equation with respect to x, we obtain

$$\frac{dT}{dx}+\frac{q}{k}x=C_1$$

Integrate the above equation with respect to x, we obtain

$$T(x)=-\frac{q}{2k}x^2+C_1x+C_2$$

Boundary Condition 1, at x = 0, T(x) = T_A (i)
Boundary Condition 2, at x = L, T(x) = T_B (ii)
ii. From Boundary Condition 1, $C_2 = T_A$
ii. From Boundary Condition 2,

$$T_B=-\frac{q}{2k}L^2+C_1L+T_A$$

$$C_1=\frac{T_B-T_A}{L}+\frac{q}{2k}L$$

$$T(x)=\left[\frac{T_B-T_A}{L}+\frac{q}{2k}(L-x)\right]x+T_A$$

$$T(x)=\left[\frac{200-100}{0.02}+\frac{10^6}{2x0.5}(0.02-x)\right]x+100$$

$$T(x)=\left[5000+10^6(0.02-x)\right]x+100$$

at x = 0.0025, T(0.0025) = 156.25°C

at $x = 0.0075$, $T(0.0075) = 231.25°C$
at $x = 0.0125$, $T(0.0125) = 256.25°C$
at $x = 0.0175$, $T(0.0175) = 231.25°C$

$kA/\delta x = (0.5*1)/0.005 = 100$ W/K
$2kA/\delta x = (2*0.5*1)/0.005 = 200$ W/K
$qA\delta x = (10^6*1*0.005) = 5000$ W
$qA\delta x/(kA/\delta x) = 5000/100 = 50$ K

The discretised equation for nodal point 2

$$2T_2 = T_1 + T_3 + 50$$

The discretised equation for nodal point 3

$$2T_3 = T_2 + T_4 + 50$$

The discretised equation for nodal point 1

$$3T_1 = T_2 + 2T_A + 50 = T_2 + 250$$

The discretised equation for nodal point 4

$$3T_4 = T_3 + 2T_B + 50 = T_3 + 450$$