Mansoura University		Pattern Recognition
Faculty of Engineering		CSE 397
Biomedical Eng. Program		Fall-2017
Full Mark is **50**		Time Allowed is **60 Minutes**
Exam is **FOUR** Questions, **ONE** page, Attempt **ALL** questions		

Q1) Explain the following terms (*i*) a pattern, (*ii*) a class, (*iii*) a classifier, (*iv*) feature space, (*v*) a decision rule, and (*vi*) a decision boundary. **[12 Points]**

Q2) Suppose that a rare disease affects 1 out of every 1,000 people in a population, and suppose that there is a good, but not perfect, test for the disease. For a person who has the disease, the test comes back positive 99% of the time and if for a person who does not have the disease the test is negative 98% of the time. A person has just tested positive; what are his chances of having the disease? If this person takes the test a second time, what is the probability that he has a disease? Comment on your results. **[12 Points]**

Q3) Suppose a pattern recognition problem has two classes ω_1, ω_2 with prior probabilities $P(\omega_1) = 3/4$ and $P(\omega_2) = 1/4$. The likelihood functions of the two classes are one-dimensional densities given in the form $p(x|\omega_i) = A_i e^{-\left(\frac{|x-\beta_i|}{2\alpha_i}\right)}$, i=1,2 and β_i is constant and $\alpha_i > 0$ **[12 Points]**

(i) Write an analytical expression for each density (i.e., find A_i)
(ii) Calculate the likelihood ratio and the posterior probability ratio.

Q4) Using the likelihood ratio, what are the decision thresholds for two class-conditional probabilities which are Gaussian in shape, with means $\mu_1 = 4$ and $\mu_2 = 10$, variances $\sigma_1^2 = 4$ and $\sigma_2^2 = 1$, and prior probabilities $P(\omega_1) = 2/3$ and $P(\omega_2) = 1/3$. **[14 Points]**

[Hint Gaussian probability density function with a mean μ and standard deviation σ is given as: $\frac{1}{\sqrt{\sigma^2 2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$]

Wish you all the best
Dr. Fahmi Khalifa

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Q1) Explain the following terms (1) a pattern, (2) a class, (3) a classifier, (4) feature space, (5) a decision rule, and (6) a decision boundary. **[12 Points, each is two points]**

Answer

- A pattern is used loosely to refer to objects that are similar in some respect.
- A class is a collection of objects that have been grouped together because they are all similar in some features.
- A classifier is some process by which objects of the same class can be distinguished (sorted).
- Feature space is the space that contains the measurements of the features that have been extracted from a set of objects to be classified. More specifically, each feature can be thought of as a measurement along some dimension, and 'n' features are contained in an n-dimensional feature space.
- A decision rule is a logical condition or set of conditions that decide which class an object belongs to.
- A decision boundary is the contour that partitions the feature space into separate classes. Objects falling on different sides of a decision boundary will belong to different classes.

Q2) Suppose that a rare disease affects 1 out of every 1,000 people in a population, and suppose that there is a good, but not perfect, test for the disease. For a person who has the disease, the test comes back positive 99% of the time and if for a person who does not have the disease the test is negative 98% of the time. A person has just tested positive; what are his chances of having the disease? If this person takes the test a second time, what is the probability that he has a disease? Comment on your results. **[15 Points]**

Solution

- Let P(A) denote the probability that a person actually has a disease
  $P(A) = .001$, $P(\bar{A})) = 1 - P(A) = 0.999)$
- Let P(B) the probability the test is positive.
- Sensitivity $P(B|A) = 0.99$, Specificity $P(\bar{B}|\bar{A}) = 0.98$
- When does the test come positive?
  In two events $\{B,A\}$ or $\{B,\bar{A}\}$ $\rightarrow\{,\}$ means joint occurrence
  Thus $P(B) = P(\{B,A\}) + P(\{B,\bar{A}\})$
  Note that $P(B|A) = \frac{P(B,A)}{P(A)} \Rightarrow P(B,A) = P(B|A)P(A)$
  Also $P(B|\bar{A}) = \frac{P(B,A)}{P(\bar{A})} \Rightarrow P(B,\bar{A}) = P(B|A)P(\bar{A})$
  $P(B) = P(B|A)P(A) + P(B|\bar{A})P(\bar{A})$
- We want to know the probability of having the disease if you test positive (the posterior probability or positive predictive value of the test), P(A|B).

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$$\text{Bayes' Rule, } P(A|B) = \frac{P(B|A).P(A)}{P(B)}$$

- Now $P(B|\bar{A})$ is the probability of testing positive given that you don't have the disease. Here, $P(B|\bar{A})=1-P(\bar{B}|\bar{A})=1-0.98=0.02$,
- Therefore, $P(B) = (0.99*0.001) + (0.02*0.999) = 0.02097$
- Returning to Bayes' Rule, $P(A|B) = 0.00099/0.02097 = 4.721*10^{-2}$ (4.721 %).

- If the person has the test for the second time $P(A) = 0.04721$ and $P(\bar{A}) = 0.9527$

The test still the same, $P(B) = (0.99*4.721*10^{-2}) + (0.02*0.9527) = 0.06578$

$P(A|B) = (0.99*0.04721)/0.06578 = 0.7104$ (71 %).

Q3) Suppose a pattern recognition problem has two classes ω_1, ω_2 with prior probabilities $P(\omega_1) = 3/4$ and $P(\omega_2) = 1/4$. The likelihood functions of the two classes are one-dimensional densities given in the form $p(x|\omega_i) = A_i e^{-\left(\frac{|x-\beta_i|}{2\alpha_i}\right)}$, $i=1,2$ and $\alpha_i > 0$

a) Write an analytical expression for each density (Find A_i)
b) Calculate the likelihood ratio and the posterior probability ratio.

Solution

Both classes have the same likelihood functions (density or pdf). From the properties of the pdf

$$\int_{-\infty}^{\infty} pdf = 1$$

$$\alpha_i > 0 \Rightarrow e^{-\left(\frac{|x-\beta|}{2\alpha}\right)} = \begin{cases} e^{-\left(\frac{x-\beta}{2\alpha}\right)} & if\ x > \beta \\ e^{\left(\frac{x-\beta}{2\alpha}\right)} & if\ x < \beta \end{cases}$$

$$\int_{-\infty}^{\infty} pdf = 1 \Rightarrow \int_{-\infty}^{\beta} Ae^{\left(\frac{x-\beta}{2\alpha}\right)} dx + \int_{\beta}^{\infty} Ae^{-\left(\frac{x-\beta}{2\alpha}\right)} dx = 1$$

$$1 = A\frac{e^{\left(\frac{x-\beta}{2\alpha}\right)}}{\frac{1}{2\alpha}}\Bigg|_{-\infty}^{\beta} + A\frac{e^{-\left(\frac{x-\beta}{2\alpha}\right)}}{-\frac{1}{2\alpha}}\Bigg|_{\beta}^{\infty}$$

$$1 = 2\alpha A[(e^0 - e^{-\infty}) - (e^{-\infty} - e^0)]$$

$$A = \frac{1}{4\alpha} \Rightarrow p(x|\omega_i) = \frac{1}{4\alpha_i}e^{\left(\frac{|x-\beta_i|}{2\alpha_i}\right)}$$

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The likelihood ratio $= \dfrac{p(x|\omega_2)}{p(x|\omega_1)} = \dfrac{\frac{1}{4\alpha_2}e^{-\left(\frac{|x-\beta_2|}{2\alpha_2}\right)}}{\frac{1}{4\alpha_1}e^{-\left(\frac{|x-\beta_1|}{2\alpha_1}\right)}} = \dfrac{\alpha_1}{\alpha_2}e^{\left[\left(\frac{|x-\beta_1|}{2\alpha_1}\right)-\left(\frac{|x-\beta_2|}{2\alpha_2}\right)\right]}$

The posterior ratio $= \dfrac{P(x|\omega_2)}{P(x|\omega_1)} = \dfrac{P(\omega_2)p(x|\omega_2)}{P(\omega_1)p(x|\omega_1)} = \left(\dfrac{0.25}{07.5}\right)\left(\dfrac{p(x|\omega_2)}{p(x|\omega_1)}\right)$

$= \left(\frac{1}{3}\right)$ x likelihood ratio

Q4) What are the decision thresholds for two class-conditional probabilities which are Gaussian in shape, with means $\mu_1 = 4$ and $\mu_2 = 10$, variances $\sigma_1^2 = 4$ and $\sigma_2^2 = 1$, and prior probabilities $P(\omega_1) = 2/3$ and $P(\omega_2) = 1/3$.

Solution: using the likelihood ratio

$$\frac{\frac{2}{3} * \frac{1}{2}e^{-\frac{1}{2*4}*(x-4)^2}}{\frac{1}{3}e^{\left(-\frac{1}{2}*(x-10)^2\right)}} = 1$$

$$\frac{e^{-\frac{1}{2*4}*(x-4)^2}}{e^{\left(-\frac{1}{2}*(x-10)^2\right)}} = 1$$

$$e^{-\frac{(x-4)^2}{8}} = e^{-\frac{(x-10)^2}{2}}$$

$$\frac{-(x-4)^2}{8} = \frac{-(x-10)^2}{2}$$

$$(x-4)^2 = 4(x-10)^2$$

$$x^2 - 8x + 16 = 4x^2 - 80x + 400$$

$$3x^2 - 72x + 384 = 0$$

Solving the quadratic gives the two roots, x = 8 and x = 16, which are the decision points.